

# EXAMPLE OF AN ORDER 16 NON SYMPLECTIC ACTION ON A K3 SURFACE

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**ABSTRACT.** We exhibit an example of a K3 surface of Picard rank 14 with a non-symplectic automorphism of order 16 which fixes a rational curve and 10 isolated points. This settles the existence problem for the last case of Al Tabbaa, Sarti and Taki's classification [1].

## 1. INTRODUCTION

Let  $X$  be a K3 surface and  $\sigma$  an automorphism of  $X$ . The action of  $\sigma$  on a volume form on  $X$  gives rise to a character  $\sigma^*$ ; when this character is non-trivial, we say that the action of  $\sigma$  is non-symplectic. Moreover, we call the action primitive if  $\sigma$  and  $\sigma^*$  have the same order. In [1], D. Al Tabbaa, A. Sarti and S. Taki classify K3 surfaces admitting a primitive non-symplectic automorphism of order 16. Their classification is complete but for the existence of a K3 surface of Picard lattice  $U(2) \oplus D_4 \oplus E_8$  with a non-symplectic automorphism of order 16 whose fixed locus would consist of  $N = 10$  isolated points and  $k = 1$  rational curve. In this brief note, we use our 'rigidity' criterion (see [3] or [4]) on symplectic or non-symplectic actions on graphs of rational curves to construct a model of such a K3 surface and automorphism.

We also give a different representation of the action with  $N = 4$  fixed points and no fixed lines found in Al Tabbaa et al. This allows us to show explicitly that both their action and our action can be distinct factorizations of a same non-symplectic automorphism of order 8.

**Legend 1.1.** *To facilitate a diagonal reading of this article, we list here the conventions used in the Figures below. Legend for Figure 1 :  $\bullet$  component of the resolution of a singularity of the sextic; for Figure 2 :  $\bullet$  component of singular fiber,  $\circ$  section or multi-section; for Figures 4 - 7:  $\bullet$  fixed curve,  $\circ$  mobile curve,  $\bullet$  stable curve,  $\blacklozenge$  fixed point.*

## 2. PREMISES

In [1], it is shown that if a K3 surface of Picard rank 14 admits a non-symplectic automorphism  $\sigma$  of order 16 with a fixed locus consisting of  $N = 10$  isolated points and  $k = 1$  fixed rational curve, then the Picard lattice of the surface is isomorphic to  $U(2) \oplus D_4 \oplus E_8$ . Moreover, to illustrate another type of action (see Section 5), the authors provide a model of such a surface:

Define  $S$  as the reducible planar sextic of equation

$$x_0 (x_0^4 x_2 + x_1^5 - 2x_1^3 x_2^2 + x_2^4 x_1) = 0.$$

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The sextic consists of a line  $L$  of equation  $x_0 = 0$  and a quintic  $C$  described by remaining factor in the above equation. The surface with Picard lattice  $U(2) \oplus D_4 \oplus E_8$  is the resolution of the double cover of the plane branched along  $S$ .

### 3. THE PICARD LATTICE

Given the Picard lattice, one can rebuild the incidence graph of smooth rational curves on a K3 surface. This was explained, and worked out explicitly for the above lattice, by S.-M. Belcastro in [2]. We represent this graph in Figure 1. Part of the rational curves on the above surface are the exceptional curves that appear when blowing up the two  $D_6$  and the  $A_1$  singularities which lie at the intersection of  $L$  and  $C$ . Given that these three configurations are disjoint, we can identify them unambiguously on the above graph – the vertices corresponding to these curves are colored in grey on the graph. Furthermore, we can now identify  $L$  on the graph: it is the lone curve which intersects all three singular configurations:

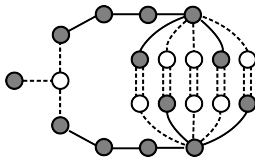
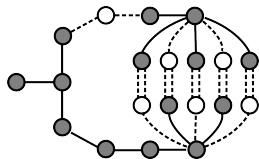
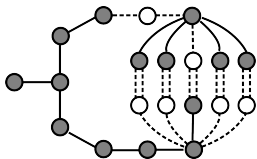


FIGURE 1. Exceptional curves coming from blowing-up along the sextic  $S$ .

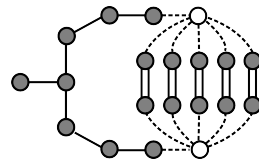
Our argument will rely heavily on the above graph. Before we move on, remark that, as in Figure 2, we can decompose the graph in different ways. These decompositions come in parallel with distinguished types of genus 1 fibrations on our surface.



(A) An  $I_0^*$  and an  $I_4^*$  fiber, five 2-sections and one section



(B) An  $I_0^*$  and an  $II^*$  fiber, six 2-sections



(C) A  $III^*$  and five  $III$  fibers, two sections

FIGURE 2. Lattice decompositions of  $U \oplus D_8 \oplus D_4 \simeq U(2) \oplus E_8 \oplus D_4$ .

### 4. BUILDING MODELS FROM STICK FIGURES

Let us start by reminding our 'rigidity' principle - which we obtain *mutatis mutandis* from [4, Lemma 8.1]:

**Philosophy 4.1.** *The action of a non-symplectic group on a configuration of rational curves is essentially determined by the action at a single point.*

More concretely, the action at two transverse rational curves is constrained by the action on the volume form, and the action at two fixed points on a given curve is constrained by the nature of automorphisms of  $\mathbb{P}^1$ . In particular, for a primitive non-symplectic automorphism of order 16, in a chain of transverse rational curves, we get the action described in Figure 3.

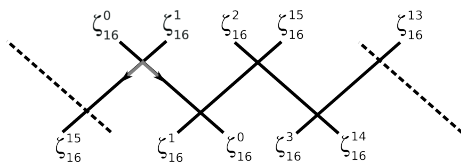


FIGURE 3. Chain of rational curves.

The annotation  $\zeta_{16}^k$  means that along that rational curve, around the point of intersection, the action looks locally like  $z \mapsto \zeta_{16}^k z$ . In accordance with the legend in the Introduction, we stylise this action as in Figure 4.

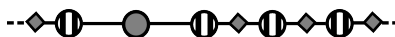


FIGURE 4. Stylized chain of rational curves.

Note that we do not mark the local action at the intersection points. This can be obtained in a trice from the observation that along the fixed curve, marked in grey, the associated action is  $z \mapsto \zeta_{16}^0 z$ .

Let us now analyze the situation when the fixed locus consists of 1 rational curve and 10 isolated points.

**Lemma 4.2.** *There is, up to isometry, a unique order 16 action on the  $U(2) \oplus D_4 \oplus E_8$  configuration of rational curves that fixes 1 rational curve and 10 points. This action is described in Figure 5.*

*Proof.* Let us first focus on the action restricted to  $L$ .

The automorphism either permutes or stabilizes the upper and lower part of the graph. Since we need 10 fixed points, the only possibility is that the upper and the lower branch are stable. From the symmetry of the graph, the curve coming from the resolution of the  $A_1$  singularity is also stable. We have thus 3 fixed points on  $L$ , which means that the action is trivial on  $L$ . Using our rigidity philosophy, we can deduce the action on the whole graph. In particular, on the rational curve corresponding to the vertices of degree 6, the action has the form  $z \mapsto \zeta_{16}^4 z$ , i.e. it is of order 4. This implies that 4 of the pairs of rational curves are permuted and 1 is stable.  $\square$

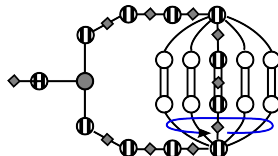


FIGURE 5. Description of the action

We are now ready to describe an explicit model of the sought after automorphism.

**4.1. A suitable Weierstrass model.** From Lemma 4.2 we see that the action preserves the  $III^* (\tilde{E}_7)$  fiber and hence the fibration described in Figure 2c. We also see that the action on the sections is of order 4, whence it is of the same order on the base. Given that the fibration has a singular fiber of type  $III^*$  and five fibers of type  $III$ , we can use the discriminantal vanishing criteria, as found in Miranda [5, Table IV.3.1], to write down the associated model:

$$\begin{cases} y^2 &= x^3 + t^3(t^4 - 1)x \\ \sigma &: (x, y, t) \mapsto (\zeta_{16}^6 x, \zeta_{16}^9 y, \zeta_{16}^4 t) \end{cases}$$

One sees directly that the action is primitive (that is the induced action on  $H^{2,0}(X, \mathbb{C})$  is also of order 16) from the explicit equation of the volume form:  $\frac{dx \wedge dt}{y}$ .

## 5. RELATION TO THE OTHER NON-SYMPLECTIC ACTION

Figure 6 schematizes the action, on the same surface as above, with  $N = 4$  isolated fixed points and  $k = 0$  fixed rational curves described by D. Al Tabbaa et al. in op. cit.

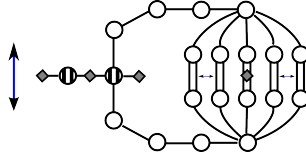


FIGURE 6. Alternative non-symplectic action as in Al Tabbaa et al.

As for the previous action, the  $\tilde{E}_7$  fiber is preserved so we can expect to have a representation of this  $N = 4$ ,  $k = 0$  automorphism as a Weierstrass model. Indeed, we have

$$\begin{cases} y^2 &= x^3 + t^3(t^4 - 1)x \\ \sigma_{AST} &: (x, y, t) \mapsto \left( \zeta_{16}^6 \frac{y^2 - x^3}{x^2}, \zeta_{16}^9 \frac{x^3 y - y^3}{x^3}, \zeta_{16}^4 t \right) \end{cases}$$

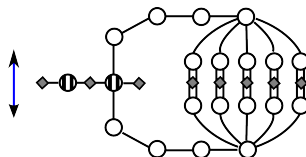
One recognizes in the action on the fibers the composition of the map  $(x, y) \mapsto (\zeta_{16}^6 x, \zeta_{16}^9 y)$  and of the affine representation of the translation by the 2 torsion section  $(0, 0) : (x, y) \mapsto (\frac{y^2 - x^3}{x^2}, \frac{x^3 y - y^3}{x^3})$ . Since the two torsion section is fixed by the automorphism of order 4, these two actions commute.

Al Tabbaa et al., showed in their work how the fixed locus of  $\sigma^2$  and  $\sigma_{AST}^2$  are of the same type (i.e.  $N_{\sigma^2} = 10$  and  $k_{\sigma^2} = 1$ ). What the Weierstrass models above show is that it is possible for the two automorphisms to be different factorizations of the same automorphism of order 8, i.e.  $\sigma_{AST}^2 = \sigma^2$ .

Finally, let us make the following observation. Through composition of  $\sigma$  and  $\sigma_{AST}^{-1}$ , we obtain a Nikulin involution (i.e. a symplectic involution):

$$\tau_{\text{symp}} := \sigma \circ \sigma_{AST}^{-1} = \left( \frac{y^2 - x^3}{x^2}, \frac{x^3 y - y^3}{x^3}, t \right)$$

And as we can see from our rigidity principle - by adding the weights of the respective actions at the germs around the fixed points - the action of  $\tau$  fixes 8 points (see Figure 7) which is exactly the number predicted in V. Nikulin's work (see [7, 6]).

FIGURE 7. Symplectic action of  $\tau$ .

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